

Introducing the Discrete Fourier Transform

Historically the (continuous) **Fourier Transform** and its close relative, the **Laplace Transform** came earlier and were regarded as being of substantially greater importance than the *Discrete Fourier Transform*, which operates on finite sequences of numbers, i.e., finite dimensional vectors, rather than on continuously defined functions. That was to be expected because the continuous transforms could be evaluated and used for significant classes of functions whereas the discrete transform required large amounts of computational effort to apply them to vectors of even moderately large dimension. All of that has changed with the advent and wide distribution of modern computing power and with the re-casting of the Discrete Fourier Transform as the Fast Fourier Transform. Modern signal processing and systems analysis rely very heavily on rapid computer processing of the discrete transform.

Understanding the Discrete Fourier Transform requires facility with the **complex extension of the standard exponential function**. If $z = x + iy$ is a complex number the exponential e^z is defined by

$$e^z = e^{x+iy} \equiv e^x (\cos y + i \sin y) \ (\equiv e^x \operatorname{cis} y),$$

where e^x is the usual exponential function evaluated at the real number x . In particular, for $x = 0$, we have the important relationship

$$e^{iy} = \cos y + i \sin y \equiv \operatorname{cis} y.$$

The rules for complex multiplication developed in the **basic material on complex arithmetic** together with the **trigonometric identities**

$$\cos(\alpha+\beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta, \sin(\alpha+\beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta,$$

show that if $w = u + iv$ then

$$\begin{aligned}
e^{z+w} &= e^{(x+u)+i(y+v)} = e^{x+u} (\cos(y+v) + i \sin(y+v)) \\
&= e^{x+u} (\cos y \cos v - \sin y \sin v + i (\sin y \cos v + \cos y \sin v)) \\
&= e^x e^u (\cos y + i \sin y)(\cos v + i \sin v) = e^z e^w.
\end{aligned}$$

Along the same lines one can also show that $(e^z)^w = e^{zw}$. Thus the exponential function has the same properties as a function of a complex variable as it does as a function of a real variable.

Example 1 What is the value of $e^{\frac{1+i\pi}{2}}$? Using the formula we have

$$e^{\frac{1+i\pi}{2}} = e^{\frac{1}{2}} e^{\frac{i\pi}{2}} = e^{\frac{1}{2}} \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right) = i\sqrt{e}.$$

We are concerned in this section with vectors whose components are certain complex numbers of modulus 1, lying on the unit circle in the complex plane. For a given value of N we define

$$\theta_0 = 0, \theta_1 = \frac{2\pi}{N}, \dots, \theta_k = \frac{2\pi k}{N}, \dots$$

These apply even if k is negative; thus $\theta_{-k} = -\theta_k$ for all integers k . For each such θ_k , which corresponds to an angle given in radian measure, we define the complex number

$$w_k = e^{i\theta_k} = \cos \theta_k + i \sin \theta_k; |w_k| = \cos^2 \theta_k + \sin^2 \theta_k = 1.$$

The w_k are evenly spaced points on the unit circle in the complex plane. The point w_0 is just 1 and the points w_k , $k = 1, 2, \dots, N-1$, rotate around the circle at equal angular intervals; for $k = N$ we have $w_N = e^{\frac{(2\pi N)i}{N}} = e^{2\pi i} = 1$ and we have come full circle; indeed, if k is any integer multiple

of N , positive or negative, the corresponding w_k will be 1. These points w_k have some very special properties. For example:

$$w_k \times w_j = e^{i \frac{2\pi k}{N}} \times e^{i \frac{2\pi j}{N}} = e^{i \frac{2\pi(k+j)}{N}} = w_{k+j};$$

$$(w_k)^j = \left(e^{i \frac{2\pi k}{N}}\right)^j = e^{i \frac{2\pi k j}{N}} = w_{kj}; \quad w_{k \pm N} = e^{i \frac{2\pi(k \pm N)}{N}} = e^{i \frac{2\pi k}{N}} e^{\pm 2\pi i} = w_k;$$

$$w_{-k} = \cos(-\theta_k) + i \sin(-\theta_k) = \cos \theta_k - i \sin \theta_k = -w_k = \overline{w_k}.$$

Next we define what we will call the *Fourier vectors* of dimension N , namely:

$$W_k = \begin{pmatrix} w_k^0 (= 1) \\ w_k \\ w_k^2 \\ \vdots \\ w_k^{N-1} \end{pmatrix}, \quad k = 0, 1, 2, \dots, N-1.$$

It should be observed that all of the components of W_0 are equal to 1 and, for every k , the first component of W_k is equal to 1. From the definition, the j -th component of W_k is w_{jk} which is the same as the k -th component of W_j , i.e., w_{kj} . Thus when we form the *Fourier matrix* of order N , which is simply the $N \times N$ matrix whose columns are the vectors W_k in the natural order:

$$\mathbf{F}_N = (W_0 \quad W_1 \quad \cdots \quad W_{N-1}),$$

we see that it has the form

$$\mathbf{F}_N = \begin{pmatrix} 1 & 1 & \cdots & 1 & \cdots & 1 \\ 1 & w_1 & \cdots & w_j & \cdots & w_{N-1} \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & w_k & \cdots & w_{jk} & \cdots & w_{(N-1)k} \\ \vdots & \vdots & & \vdots & & \vdots \\ 1 & w_{N-1} & \cdots & w_{j(N-1)} & \cdots & w_{(N-1)^2} \end{pmatrix};$$

clearly, from the properties of the w_k noted earlier, it is a *symmetric* matrix: $\mathbf{F}^T = \mathbf{F}$.

Example 2 For $N = 6$ we have

$$\mathbf{F}_6 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} + i \frac{\sqrt{3}}{2} & -\frac{1}{2} + i \frac{\sqrt{3}}{2} & -1 & -\frac{1}{2} - i \frac{\sqrt{3}}{2} & \frac{1}{2} - i \frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} + i \frac{\sqrt{3}}{2} & -\frac{1}{2} - i \frac{\sqrt{3}}{2} & 1 & -\frac{1}{2} + i \frac{\sqrt{3}}{2} & -\frac{1}{2} - i \frac{\sqrt{3}}{2} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{2} - i \frac{\sqrt{3}}{2} & -\frac{1}{2} + i \frac{\sqrt{3}}{2} & 1 & -\frac{1}{2} - i \frac{\sqrt{3}}{2} & -\frac{1}{2} + i \frac{\sqrt{3}}{2} \\ 1 & \frac{1}{2} - i \frac{\sqrt{3}}{2} & -\frac{1}{2} - i \frac{\sqrt{3}}{2} & -1 & -\frac{1}{2} + i \frac{\sqrt{3}}{2} & \frac{1}{2} + i \frac{\sqrt{3}}{2} \end{pmatrix}.$$

We will suppress the subscript N in \mathbf{F}_N henceforth whenever the dimensional context is unambiguous.

To introduce the discrete Fourier transform of order N , let us suppose that Z is an N -dimensional complex vector. Since there are N of the vectors W_k , it is natural to suspect that they might form a basis for E^N , in which case there would be unique coefficients c_0, c_1, \dots, c_{N-1} such that

$$Z = c_0 W_0 + c_1 W_1 + \dots + c_{N-1} W_{N-1};$$

that is, taking the matrix \mathbf{F} as defined above and taking C to be the N -dimensional vector whose components are the c_k ,

$$Z = \mathbf{F} C.$$

But how do we find C ? Clearly if \mathbf{F} is a nonsingular matrix, so that the inverse matrix \mathbf{F}^{-1} exists, we need only multiply the previous equation on the left by \mathbf{F}^{-1} to obtain

$$\mathbf{F}^{-1} Z = \mathbf{F}^{-1} \mathbf{F} C = C$$

so that we have

$$C = \mathbf{F}^{-1} Z.$$

But how do we find, assuming it exists, the inverse matrix? It is here that the real beauty and simplicity of the discrete Fourier transform manifests itself. For, as we will see, we have

$$\mathbf{F}^{-1} = \frac{1}{N} \overline{\mathbf{F}};$$

i.e., the inverse is simply equal to $\frac{1}{N}$ times the conjugate of the matrix \mathbf{F} .

Thus in the case of \mathbf{F}_6 shown above we have

$$\mathbf{F}_6^{-1} = \frac{1}{6} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} - i \frac{\sqrt{3}}{2} & -\frac{1}{2} - i \frac{\sqrt{3}}{2} & -1 & -\frac{1}{2} + i \frac{\sqrt{3}}{2} & \frac{1}{2} + i \frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} - i \frac{\sqrt{3}}{2} & -\frac{1}{2} + i \frac{\sqrt{3}}{2} & 1 & -\frac{1}{2} - i \frac{\sqrt{3}}{2} & -\frac{1}{2} + i \frac{\sqrt{3}}{2} \\ 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -\frac{1}{2} + i \frac{\sqrt{3}}{2} & -\frac{1}{2} - i \frac{\sqrt{3}}{2} & 1 & -\frac{1}{2} + i \frac{\sqrt{3}}{2} & -\frac{1}{2} - i \frac{\sqrt{3}}{2} \\ 1 & \frac{1}{2} + i \frac{\sqrt{3}}{2} & -\frac{1}{2} + i \frac{\sqrt{3}}{2} & -1 & -\frac{1}{2} - i \frac{\sqrt{3}}{2} & \frac{1}{2} - i \frac{\sqrt{3}}{2} \end{pmatrix}.$$

Before going on to see why this is the case, we offer the following

Definition The vector C given by $C = \mathbf{F}^{-1} Z$ is the *discrete Fourier transform* of the vector Z . Conversely, if C is given, $Z = \mathbf{F} C$ is the *inverse discrete Fourier transform* of C .

Each of the vectors W_k has *frequency* k in the sense that as j runs through the values from 0 to $N - 1$ the complex numbers w_{kj} rotate around the unit circle in the complex plane a total of k times, starting at $w_{k0} = 1$, always returning to the value 1 when $j = N$; the component w_{kN} is not included in W_k because it would be superfluous. When the vector Z is expressed in the form $Z = \mathbf{F} C$, the components of the Fourier transform vector C give the frequency content of the vector Z ; if we write c_k in polar form:

$$c_k = \rho_k e^{i\phi_k}, \quad \rho_k = \sqrt{(Re c_k)^2 + (Im c_k)^2}, \quad \phi_k = \tan^{-1}\left(\frac{Im c_k}{Re c_k}\right),$$

the absolute value ρ_k indicates the *amplitude* of the “ k -frequency” content of Z while ϕ_k indicates its *phase*:

$$c_k w_{kj} = \rho_k e^{i\phi_k} e^{i\frac{2\pi kj}{N}} = \rho_k e^{i\left(\frac{2\pi kj}{N} + \phi_k\right)}.$$

We will denote the discrete Fourier transform of the vector Z by $\mathcal{F}(Z)$; thus what we are saying is

$$\mathcal{F}(Z) = \mathbf{F}^{-1}(Z) = \frac{1}{N} \overline{\mathbf{F}}(Z).$$

Now we proceed to the demonstration of the existence and form of \mathbf{F}^{-1} as claimed above.

Proposition 1 *The vectors W_k , $k = 0, 1, 2, \dots, N - 1$ have the property*

$$\langle W_k, W_j \rangle = \begin{cases} N, & j = k; \\ 0, & j \neq k \end{cases},$$

i.e., they are mutually orthogonal and all have norm equal to \sqrt{N} .

Proof We recall the formula for the sum of a **geometric progression**; for any (real or complex) number a :

$$1 + a + a^2 + \dots + a^m = \frac{a^{m+1} - 1}{a - 1}.$$

Then we compute the complex inner product, in E^N , of W_k and W_j :

$$\begin{aligned} \langle W_k, W_j \rangle &= \sum_{\ell=0}^{N-1} w_{k\ell} \overline{w_{j\ell}} = \sum_{\ell=0}^{N-1} w_{k\ell} w_{-j\ell} \\ &= \sum_{\ell=0}^{N-1} w_{(k-j)\ell} = \sum_{\ell=0}^{N-1} w_{k-j}^\ell. \end{aligned}$$

Taking $a = w_{k-j}$ and $m = N - 1$ in the formula for the sum of a geometric progression, we now have, for $k \neq j$,

$$\langle W_k, W_j \rangle = \frac{w_{k-j}^N - 1}{w_{k-j} - 1} = \frac{w_{(k-j)N} - 1}{w_{k-j} - 1} = 0$$

because $w_{(k-j)N} = 1$. On the other hand, when $k = j$ we have

$$\langle W_k, W_k \rangle = \sum_{\ell=0}^{N-1} w_{(k-k)\ell} = \sum_{\ell=0}^{N-1} w_0 = \sum_{\ell=0}^{N-1} 1 = N,$$

which completes the proof of the proposition. We can summarize this by saying that the W_k , $k = 0, 1, 2, \dots, N - 1$ form an orthogonal system in the space E^N of N -dimensional complex vectors.

Theorem 1 *The inverse of the Fourier matrix \mathbf{F} is given by $\mathbf{F}^{-1} = \frac{1}{N} \bar{\mathbf{F}}$.*

Proof Because the matrix \mathbf{F} is symmetric, $\bar{\mathbf{F}}$ is also symmetric and we can write

$$\bar{\mathbf{F}} = \begin{pmatrix} W_0^* \\ \vdots \\ W_k^* \\ \vdots \\ W_{N-1}^* \end{pmatrix},$$

where, as earlier, the superscript * indicates the conjugate transpose of the column vector W_k ; the row vector whose components are the complex conjugates of the components of the k -th column, W_k , of the Fourier matrix \mathbf{F} . Then the rules of matrix multiplication give

$$\bar{\mathbf{F}} \mathbf{F} = \begin{pmatrix} W_0^* \\ \vdots \\ W_k^* \\ \vdots \\ W_{N-1}^* \end{pmatrix} (W_0 \ \cdots \ W_j \ \cdots \ W_{N-1})$$

$$= \left(\overline{\langle W_k, W_j \rangle} \right),$$

the last symbol indicating the $N \times N$ matrix whose k, j -th entry is $\overline{\langle W_k, W_j \rangle} = \langle W_j, W_k \rangle$. But the preceding proposition shows that

$$\langle W_j, W_k \rangle = \begin{cases} N, & j = k; \\ 0, & j \neq k \end{cases},$$

in other words, $\bar{\mathbf{F}}\mathbf{F} = N\mathbf{I}_N$, where \mathbf{I}_N denotes the $N \times N$ identity matrix. Dividing both sides by N we have the result in the statement of the theorem and the proof is complete.

Another way in which the result of this theorem can be stated is the following. The result as stated in the theorem, applied to the k -th component, shows that

$$c_k = (\mathcal{F}(Z))_k = \frac{1}{N} W_k^* Z = \frac{1}{N} \langle Z, W_k \rangle.$$

Example 3 Suppose we take $Z = (1 \ 1 \ \cdots \ 1)$. Then

$$\begin{aligned} (\mathcal{F}(Z))_k &= \langle Z, W_k \rangle = \frac{1}{N} \sum_{j=0}^{N-1} 1 \times e^{-i \frac{2\pi k j}{N}} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \left(e^{-i \frac{2\pi k}{N}} \right)^j. \end{aligned}$$

If $k = 0$ this is just $\frac{1}{N} \sum_{j=0}^{N-1} 1 = 1$. On the other hand, for $k \neq 0$ the formula for the sum of a geometric progression gives

$$(\mathcal{F}(Z))_k = \frac{1}{N} \frac{\left(e^{-i \frac{2\pi k}{N}} \right)^N - 1}{e^{-i \frac{2\pi k}{N}} - 1} = 0.$$

So the discrete Fourier transform of $Z = (1 \ 1 \ \cdots \ 1)$ is $\mathcal{F}(Z) = (1 \ 0 \ \cdots \ 0)$.

Example 4 Consider a slightly more complicated case:

$$Z = (0 \ 1 \ 2 \ \cdots \ N-1), \text{ i.e., } z_j = j.$$

Here for $k \neq 0$ we have

$$\begin{aligned} c_k &= \frac{1}{N} \sum_{j=0}^{N-1} j e^{-i \frac{2\pi k j}{N}} = \frac{1}{N} \sum_{j=0}^{N-1} j w_{-kj} \\ &= \frac{1}{N} \sum_{j=0}^{N-1} \left(\sum_{\ell=0}^{j-1} 1 \right) w_{-kj} = \frac{1}{N} \sum_{\ell=0}^{N-1} \left(\sum_{j=\ell+1}^{N-1} (w_{-k})^j \right) \\ &= \frac{1}{N} \sum_{\ell=0}^{N-1} (w_{-k})^{\ell+1} \sum_{j=0}^{N-1-(\ell+1)} (w_{-k})^j \\ &= \frac{1}{N} \sum_{\ell=0}^{N-1} (w_{-k})^{\ell+1} \frac{(w_{-k})^{N-(\ell+1)} - 1}{w_{-k} - 1} \\ &= \frac{1}{N} \frac{1}{w_{-k} - 1} \sum_{\ell=0}^{N-1} \left(1 - (w_{-k})^{\ell+1} \right), \end{aligned}$$

the last being true because $w_{-k}^N = 1$. But $\sum_{\ell=0}^{N-1} (w_{-k}^{\ell+1}) = 0$. On the other hand $\sum_{\ell=0}^{N-1} 1 = N$. So we have the result

$$c_k = \frac{1}{w_{-k} - 1}.$$

For $k = 0$, since $w_0 = 1$, we have

$$c_0 = \frac{1}{N} \sum_{j=0}^{N-1} j (w_0)^j = \frac{1}{N} \sum_{j=0}^{N-1} j = \frac{1}{N} \frac{N(N-1)}{2} = \frac{N-1}{2}.$$

We see therefore that for $Z = (0 \ 1 \ 2 \ \cdots \ N-1)$ we have

$$\mathcal{F}(Z) = \left(\frac{N-1}{2} \quad \frac{1}{w_1-1} \quad \cdots \quad \frac{1}{w_{N-1}-1} \right).$$